Ontario Hydro

91602
rev. 82-10

COMPLIMENTARY COPY
Please Credit Ontario Hydro Research if
Reproduced in Whole or in Part

ontario hydro
research division

AN ANALYSIS OF OPTIMUM MAINTENANCE POLICIES

Report No 85-90-K

N. Ramani
Engineer
Reliability & Statistics Section
Operations Research Department

ABSTRACT

This study analyzes the characteristics of optimum maintenance policies. The classical age-based maintenance model is solved using the semi-Markov approach. Next, the effect of the model parameters on optimum maintenance intervals, the relative frequencies of repair and maintenance events and the corresponding unavailabilities are computed. Extensions to the classical model with similar solution formats are reviewed. An optimum maintenance model based on maintenance dependent repair durations is proposed and the resulting optimal policy is compared with that using the classical model. In future studies, these models will be applied to power system components.

job	file	date	report no.
740672-615-022	813.07	May 3, 1985	85-90-K

TABLE OF CONTENTS

		Page
1.	INTRODUCTION	1
2.	THE CLASSICAL AGE-BASED MAINTENANCE MODEL	1
	2.1 The Solution of the Model	1
	2.2 Some Properties of the Optimal Solution	5
3.	CHARACTERISTICS OF THE OPTIMUM POLICY	7
	3.1 Variation of the Unavailability With the Maintenance Interval	7
	3.2 Variation of T_{opt} with the ratio m_R/m_M	7
	3.3 Variation of the Optimal Ratio of Repair and Maintenance Events, and of the Corresponding Unavailabilities, with the Ratio m_R/m_M	9
4.	OTHER OPTIMUM MAINTENANCE MODELS WITH SIMILAR SOLUTIONS	10
	4.1 Minimizing the Cost Rate	13
	4.2 Maximizing Profit and Earning Rates	13
	4.3 Minimal Repair	14
	4.4 Imperfect Preventive Maintenance	14
5.	A MODEL FOR MAINTENANCE DEPENDENT REPAIR DURATIONS	15
	5.1 The Model	15
	5.2 An Example	15
6.	CONCLUSIONS	16
	REFERENCES	18
	APPENDIX A - SEMI-MARKOV PROCESSES	19
	APPENDIX B - SUM-OF-EXPONENTIALS DISTRIBUTIONS	22
	DISTRIBUTION	last page

91601
new 82-10

ontario hydro
research division

To Mr. F.J. Kee
Director of Research

AN ANALYSIS OF OPTIMUM MAINTENANCE POLICIES

1. INTRODUCTION

In recent years, optimum maintenance policies for systems subject to stochastic failures have attracted considerable attention in the literature/2,10/. The objectives of this study are to summarize the theoretical background for the basic model, discuss the features of optimum systems and study extensions of the basic model.

In power system reliability studies, the use of Markov models for analyzing the behaviour of stochastic processes is well known. However the implicit assumption, that the time to failure of the system under study is exponentially distributed, means that the method cannot be used to determine optimum maintenance policies. This is because preventive maintenance implies a deteriorating system, and an exponential failure distribution represents a non-deteriorating system. A semi-Markov approach can handle a class of stochastic systems with non-exponential distributions.

In this study, a solution of the well-known age-based maintenance model is derived using a semi-Markov approach. The effects of the model parameters on the results are then discussed. This is followed by a summary of more complex models whose solutions have formats similar to that of the basic model. An optimum maintenance model where the repair duration is a function of the maintenance interval is proposed. In an Appendix, the use of sum-of-exponentials distributions for the time to failure is discussed.

2. THE CLASSICAL AGE-BASED MAINTENANCE MODEL

2.1 The Solution of the Model

This model was first discussed by Barlow and Hunter/1/ using renewal theory concepts and has been studied in detail in the literature. The following derivation using the semi-Markov approach is probably more appealing to engineers used to the state space approach. Appendix A gives a brief review of the semi-Markov approach.

The system under study, when not under preventive maintenance, is assumed to be in one of two states: working or failed. It is assumed that after preventive maintenance or after a repair following a failure, the system returns to 'as good as new' conditions. That is, time is reset to zero, when the system is returned to service after maintenance or repair. Figure 1 shows the system representation.

If the distributions of the time to failure, the time before preventive maintenance, the repair duration and the maintenance duration are all exponential, the system can be analyzed using a Markov model. In the general case, when the exponential assumption does not hold, each of the states, Operating (state 1), Repair (state 2) and Maintenance (state 3), can still be considered as a regeneration point and the semi-Markov approach can be used to study the behaviour of the system.

The set of successive states of the system, X_n, $n = 0, 1, \ldots$, forms a Markov chain because X_n depends on X_{n-1} alone. The index indicates the moment of a change in state. In the classical age-based maintenance policy, the system undergoes preventive maintenance at age T if no failure occurs in the period 0 to T. Figure 1 shows all the transition probabilities of the Markov chain under this policy. The probability of transition from state 1 to state 2 in one step, is the probability that the unit fails before time T, ie, is equal to F(T) where the function F(t) is the cumulative distribution of the time to failure. The probability of transition from state 1 to state 3 in one step, is the probability that the unit will survive until time T, and is equal to $\bar{F}(T) = 1 - F(T)$.

The transition probability matrix of the Markov chain is given by

$$[P] = \begin{bmatrix} 0 & F(T) & \bar{F}(T) \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad (1)$$

[P] is the matrix of one-step probabilities, ie, the (i,j)th element $p_{ij} = P[X_n=j \mid X_{n-1}=i]$. As described in Appendix A, the steady-state probabilities π_i of the Markov chain are obtained by solving the equation

$$[\pi]\,[P] = [\pi] \qquad (2)$$

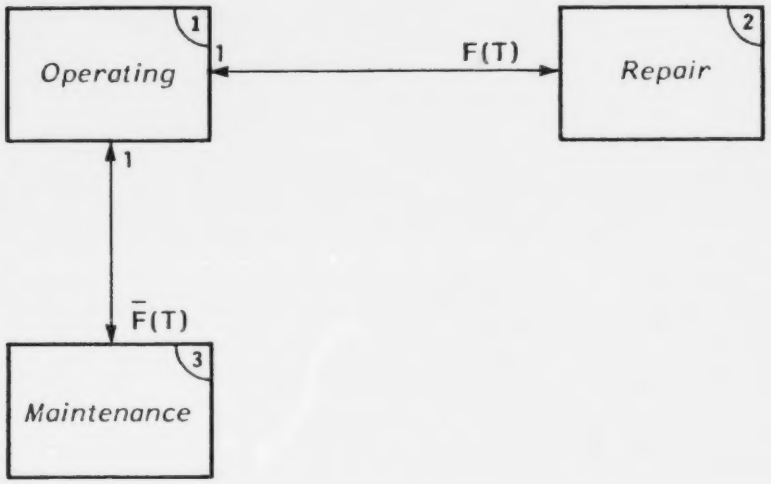


FIGURE 1
SYSTEM WITH AN AGE BASED MAINTENANCE POLICY

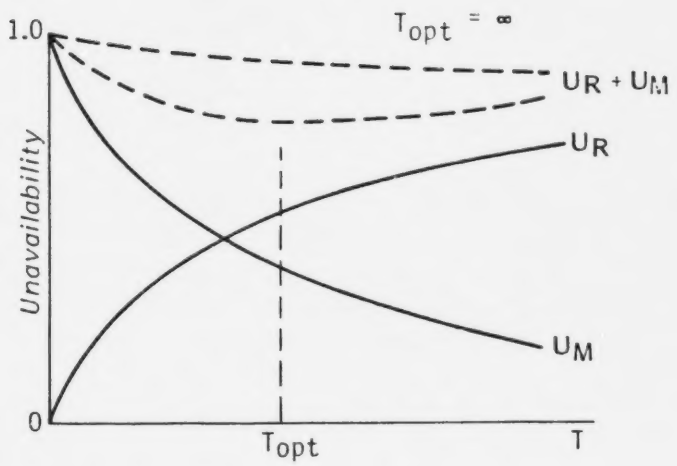


FIGURE 2
VARIATION OF UNAVAILABILITY WITH
THE MAINTENANCE INTERVAL *T*

It can easily be shown that

$$\pi_1 = 0.5 \tag{3}$$
$$\pi_2 = 0.5\ F(T) \tag{4}$$
$$\pi_3 = 0.5\ \bar{F}(T) \tag{5}$$

Now, a stochastic process z(t), defined such that z(t) = i if the system is in state i at time t, is a semi-Markov process. The above-mentioned Markov chain is said to be 'embedded' in the process. The steady-state probabilities of the process, ie of the system under study, can be obtained using Equation (A.6) of Appendix A from the above steady state probabilties of the embedded Markov chain and the mean residence times in the three states.

The mean residence time in state 1 is given by $\int_0^T \bar{F}(t)\ dt$. In states 2 and 3, the mean residence times are simply the mean times to perform the repair (m_R) and maintenance (m_M) respectively.

Thus, a very interesting and somewhat surprising aspect of the age-based maintenance policy is immediately obvious. The optimum policy, which is a function of the steady-state probabilities, is independent of the probability distributions of the repair and maintenance times - only the mean durations are of interest.

The steady-state availability of the system is given by

$$A(T) = \frac{\int_0^T \bar{F}(t)\ dt}{\int_0^T \bar{F}(t)dt + m_R\ F(T) + m_M\ \bar{F}(T)} \tag{6}$$

The unavailabilities due to repair and preventive maintenance respectively are given by

$$U_R(T) = \frac{m_R\ F(T)}{\int_0^T \bar{F}(t)dt + m_R\ F(T) + m_M\ \bar{F}(T)} \tag{7}$$

and

$$U_M(T) = \frac{m_M \bar{F}(T)}{\int_0^T \bar{F}(t)dt + m_R F(T) + m_M \bar{F}(T)} \qquad (8)$$

In this model, the optimum policy is defined to be the one which maximizes the steady-state availability. The optimum maintenance interval $T = T_{opt}$ can thus be obtained by equating the derivative of A(T) with respect to T to zero and is given by the solution of the integral equation

$$h(T) \int_0^T \bar{F}(t)dt - F(T) = m_M/(m_R - m_M) \qquad (9)$$

where h(t) is the hazard function associated with F(t).

The different definitions of the steady-state probabilities of the embedded Markov chain (equations 3,4 and 5) and those of the semi-Markov process (equations 6,7 and 8) deserve a closer look. In the former case, there is no time parameter involved, and the experiment consists of observing the states of the system only at instances of changes in states and counting the number of visits to each of the states. Therefore, the long-run probability of finding the embedded Markov chain in state i is given by

$$\pi_i = \lim_{n\to\infty} P[x_n=i] = \lim_{n\to\infty} \frac{n_i}{n}$$

where x_n is the state of the system after the nth state change and n_i the number of visits to state i occurring during the same n changes of state. The latter probabilities refer to the stochastic process z(t) defined as the state of the system at time t; ie, the long-run probability of finding the system in state i is

$$p_i = \lim_{t\to\infty} P[z(t)=i] = \lim_{t\to\infty} \frac{t_i}{t}$$

where t_i is the total time spent in state i up to time t.

2.2 Some Properties of the Optimal Solution

Some interesting aspects of the optimum policy, discussed in the literature, are listed below. Most of these comments apply specifically to the above policy of maximizing the availability, and thus cost considerations are not included. However, similar conclusions can be drawn when other policies, such as minimizing the cost, are considered.

- If there is no solution to equation 9, $T_{opt} = \infty$. That is, preventive maintenance is not efficient and no maintenance should be performed.

- If the hazard rate is constant or decreasing, there is no solution to equation 9, ie, $T_{opt} = \infty$. This is intuitively clear, because in this case, the unit either is unchanged or improves with time, and thus replacing an unfailed unit with a new unit is not efficient. Examples include the exponential distribution and the Weibull and Gamma distributions with shape factors less than 1.

- $(m_R/m_M) > 1$ is a necessary condition for a solution to equation 9 to exist, ie, for preventive maintenance to be effective. This is again intuitively clear, because, there is no point in preventive maintenance if it takes longer than a repair, when both the operations bring the unit to the same (as good as new) condition. Note that if the optimal policy is chosen to minimize the cost per unit time, the corresponding necessary condition is for the mean maintenance cost to be lower than the mean repair cost.

- If the hazard rate is monotonously increasing, the left hand side of equation 9 is also monotonously increasing. Thus there is at most one solution. In the rest of this report, it is assumed that the hazard rate is monotonously increasing, ie the unit continuously deteriorates with time.

- A unique solution to equation 9 exists if

$$m_o\, h(\infty) > m_M/(m_R - m_M)$$

  where m_o is the mean time to failure of the unmaintained system.

- A unique solution exists if $h(t) \to \infty$ as $t \to \infty$. This follows from the above remark.

- $(m_R/m_M) > 1$ is not a sufficient condition for a solution to exist even when h(t) is monotonously increasing, if $h(\infty)$ is finite. For example, for a gamma distribution, there is a minimum value of (m_R/m_M), greater than 1, below which $T_{opt} = \infty$.

- $T_{opt} > m_o m_M/m_R$.

- As already discussed, the results are independent of the distributions of the repair and maintenance times.

3. CHARACTERISTICS OF THE OPTIMUM POLICY

This section studies some of the properties of the optimum policy using some simple examples.

3.1 Variation of the Unavailability With the Maintenance Interval

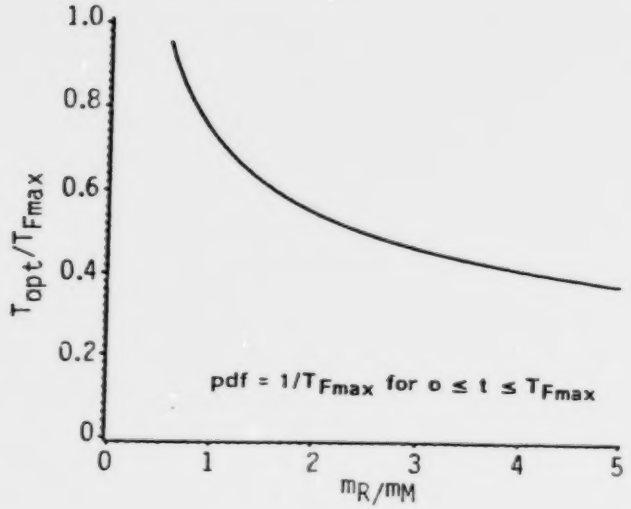
Optimum maintenance policies always involve compromises between two or more conflicting factors. In this case, the conflicting factors are (a) the unavailability due to repair following a failure which increases with the maintenance interval T, and (b) the unavailability due to preventive maintenance, which decreases with increasing maintenance interval. Equations (7) and (8) can be used to compute the unavailabilities and Figure 2 sketches them as functions of T for a typical (say, Weibull) failure distribution. U_R, the unavailability due to repair increases from 0 for $T = 0$, to $m_R/(m_O + m_R)$ for $T = \infty$. U_M, the unavailability due to preventive maintenance, decreases from a value of 1 for $T = 0$, to 0 for $T = \infty$. ($T = 0$ means that the unit is continuously under preventive maintenance and thus $U_M = 1$ and $U_R = 0$. If $T = \infty$, there is no preventive maintenance and thus $U_M = 0$ and U_R is the unavailability in a standard two-state model).

The total unavailability $U_R + U_M$ can follow either one of the dotted lines shown in Figure 2. As mentioned earlier, it has at most one minimum when the hazard rate is monotonously increasing. That is, either there is a unique finite value for T_{opt}, or $U_R + U_M$ decreases continuously with increasing T and $T_{opt} = \infty$.

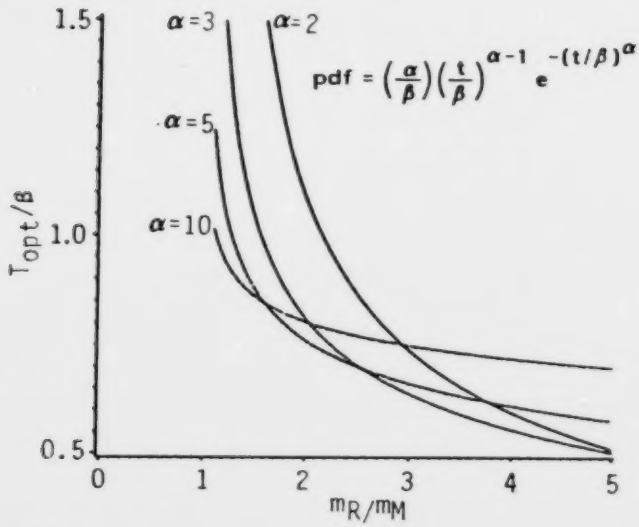
3.2 Variation of T_{opt} with the Ratio m_R/m_M

As can be seen from equation (9), the optimum maintenance interval is a function of the ratio m_R/m_M and not the actual values of m_R and m_M. The variation of T_{opt} with m_R/m_M has been studied in the literature and is included here for the sake of completeness. Figures 3(a), 3(b) and 3(c) show the variation of T_{opt} with m_R/m_M for the uniform, Weibull and gamma distributions respectively of the time to failure.

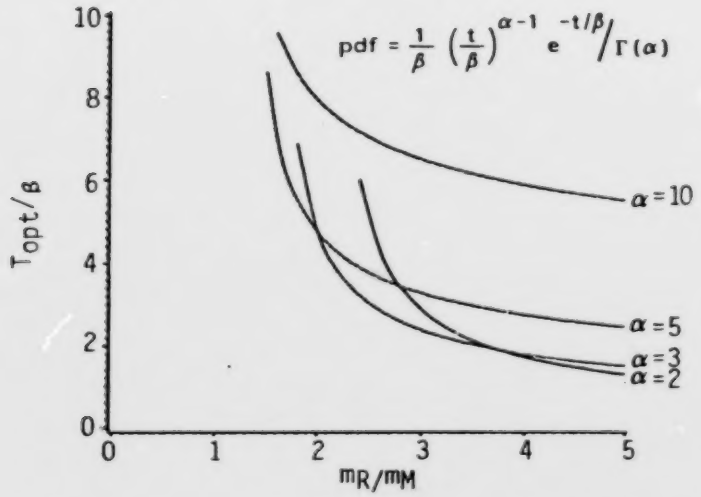
In these calculations, the time axis has been normalized by the maximum time to failure for the uniform distribution, and the scale factors for the Weibull and gamma distributions. The actual values of the optimum maintenance interval can be obtained from the plots by multiplying the values thereof by the normalizing factor. For example, consider the case where the mean time to failure of an equipment is 1000 h. If the time to



(a)
Uniform Distribution



(b)
Weibull Distribution



(c)
Gamma Distibution

FIGURE 3

OPTIMUM MAINTENANCE INTERVAL VERSUS
RATIO OF REPAIR TO MAINTENANCE DURATION

failure is given by a Weibull distribution with a shape factor of 2 (assumed to be obtained using standard statistical methods), the scale factor can be easily computed to be equal to 1128 h. That is, the distribution of the time to failure is given by

$$f(t) = (2/t)\ (t/1128)^2\ e^{-(t/1128)^2}$$

If the mean repair duration is 4 times the mean maintenance duration (ie, $m_R/m_M = 4$), one can obtain the normalized optimum maintenance interval from Figure 4 as 0.595. The optimum interval to be used is therefore 1128 x 0.595 = 670 h.

The importance of selecting the correct distribution for the time to failure can be demonstrated by using the same example. If instead of Weibull, one assumes that the time to failure is given by a gamma distribution with the same mean and variance as before, one can show that the shape factor = 3.66 and the scale factor = 273.3 h. The distribution of the time to failure then is given by

$$f(t) = 273.3\ (t/273.3)^{2.66}\ e^{-(t/273.3)}/\ \Gamma(3.66)$$

For a shape factor of 3.66, normalized T_{opt} = 2.076 for a gamma distribution. Therefore, the optimum maintenance interval is = 273.3 x 2.076 = 567 h, almost 15% smaller than before.

As shown in Figure 3(a), for the uniform distribution, T_{opt} is close to 1 for small values of m_R/m_M (as mentioned earlier, one needs $m_R/m_M > 1$ for preventive maintenance to be efficient) and decreases slowly as the ratio increases. It reaches a value of 0.5 when the ratio becomes 5. For Weibull and gamma distributions, the optimum estimates for several values of the shape parameter are shown in Figures 3(b) and 3(c). As m_R/m_M increases, T_{opt} first decreases rapidly, but for further increases, the curve tends to flatten out. For the Weibull distribution, a finite value of T_{opt} always exists because the hazard rate increases to ∞ with time. However, for the gamma distribution, for small values of m_R/m_M, T_{opt} is infinite, ie, no preventive maintenance should be carried out.

3.3 Variation of the Optimal Ratio of Repair and Maintenance Events, and of the Corresponding Unavailabilities, with the Ratio m_R/m_M

It is instructive to study the predicted number of repair and maintenance events when the optimal policy is used. Although the expected number of repair and maintenance events are functions of the actual values of m_R and m_M, it can be shown

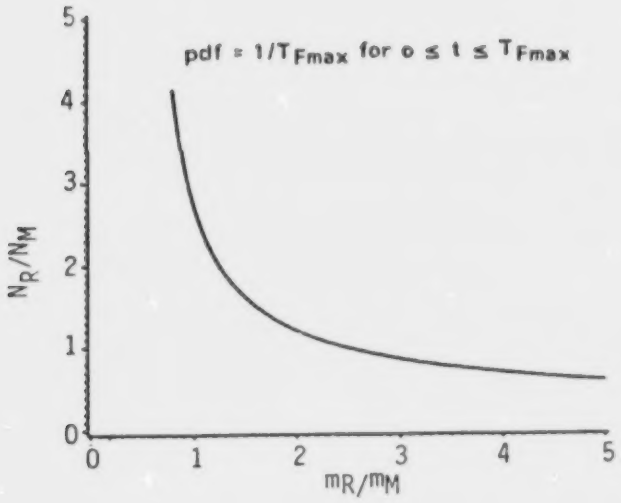
that the ratio of the expected number of repair events to the expected number of maintenance events, N_R/N_M, is a function only of the ratio m_R/m_M. The ratio N_R/N_M is plotted in Figures 4(a), 4(b) and 4(c) for the three failure distributions mentioned earlier. Note that this ratio is also independent of the scale factor of the distribution of time to failure.

The results may need some explanation. Often in practice, the number of maintenance events far exceed the number of failures. However, this is not always so in the optimum policies. The ratio N_R/N_M is large for small values of m_R/m_M and decreases as m_R/m_M increases (ie, maintenance is scheduled more often, as maintenance duration decreases relative to repair duration). For the uniform distribution, the ratio is less than 1, ie, number of maintenance events exceed number of repair events, only for $m_R/m_M > 5$. For the Weibull distribution when the shape factor is 2, the ratio N_R/N_M is less than 1 for $m_R/m_M > 2.5$. For larger shape factors, the hazard rate increases more rapidly with time, resulting in a decrease in the value of m_R/m_M at which the number of maintenance events exceed the number of repair events (ie, the ratio N_R/N_M falls below 1.0). For the gamma distribution, the results are similar to those for the Weibull except that the corresponding values of m_R/m_M at which the ratio N_R/N_M falls below 1.0 are larger.

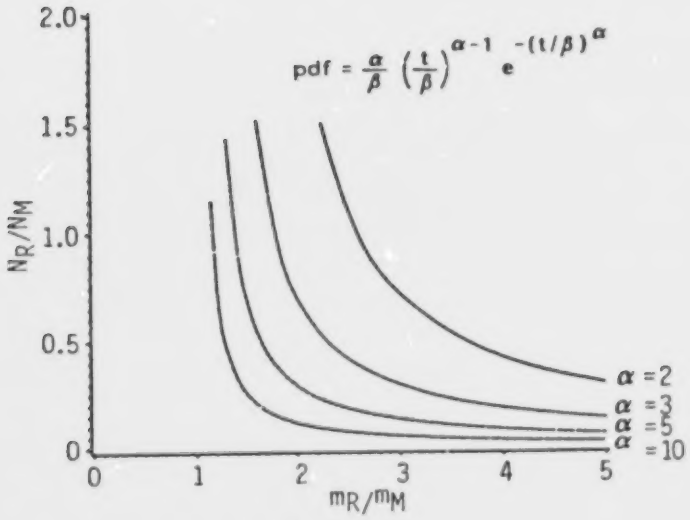
The ratio of the unavailabilities U_R and U_M (equations 7 and 8 respectively) is another non-dimensional quantity which can provide some insight into the characteristics of the optimum policy. Again, U_R and U_M depend upon the actual values of the mean repair and maintenance times, but the ratio U_R/U_M is a function of the ratio m_R/m_M alone. This ratio U_R/U_M is also independent of the scale factors of the distributions and is plotted in Figures 5(a), 5(b) and 5(c) for the three distributions. It is interesting to note that the ratio U_R/U_M is greater than 1 in most cases. For the uniform distribution, it is almost always above 5.0. For the Weibull and gamma distributions, it is less than 1.0 only for high values of the shape factor.

4. OTHER OPTIMUM MAINTENANCE MODELS WITH SIMILAR SOLUTIONS

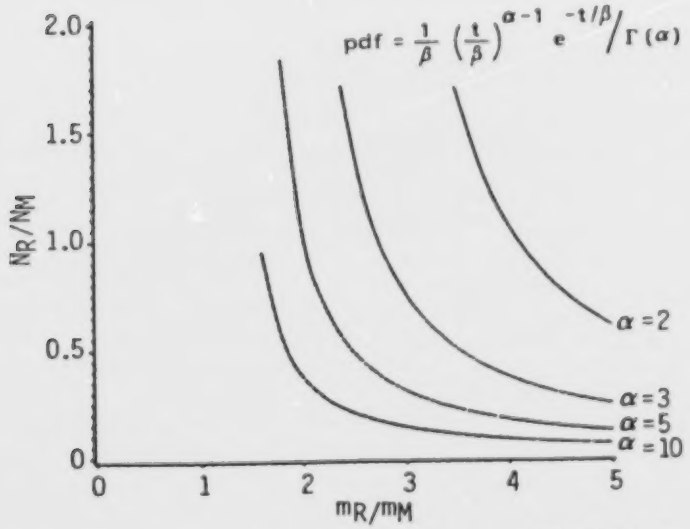
A variety of modifications and extensions to the basic policy have been reported in the literature. It is interesting to note that several of these have solution formats similar to the classical model studied above. The purpose of this section is to summarize such models.



(a)
Uniform Distribution



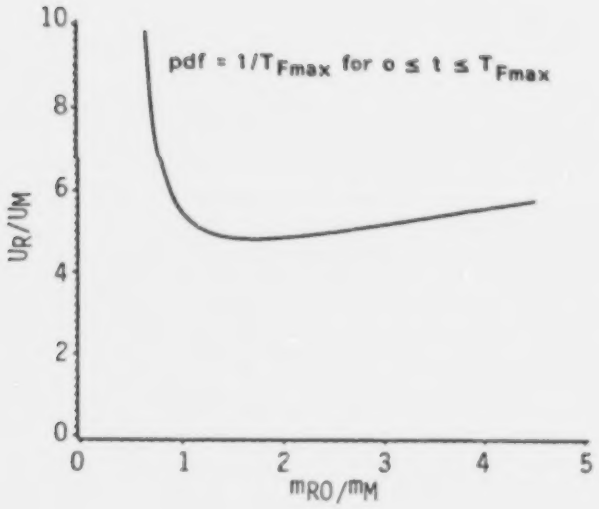
(b)
Weibull Distribution



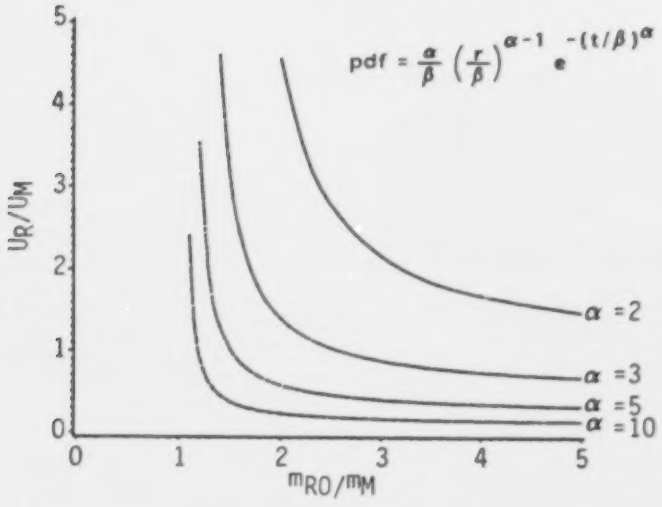
(c)
Gamma Distribution

FIGURE 4

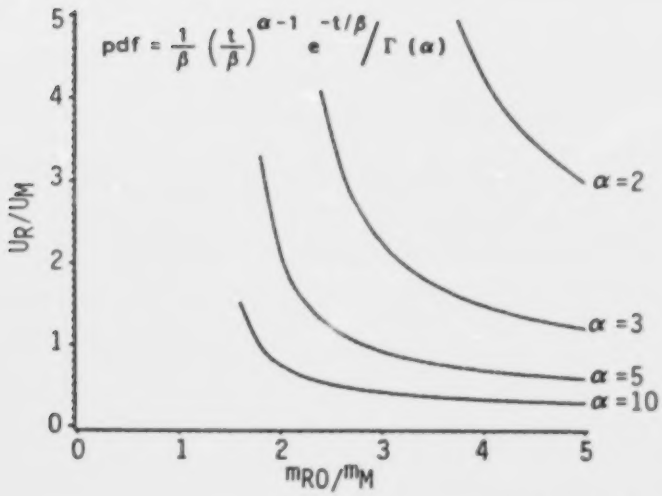
RATIO OF REPAIR TO MAINTENANCE NUMBER OF EVENTS VERSUS
RATIO OF REPAIR TO MAINTENANCE DURATION
UNDER OPTIMUM MAINTENANCE POLICY



(a)
Uniform Distribution



(b)
Weibull Distribution



(c)
Gamma Distribution

FIGURE 5

RATIO OF REPAIR TO MAINTENANCE UNAVAILABILITY VERSUS
RATIO OF REPAIR TO MAINTENANCE DURATION
UNDER OPTIMUM MAINTENANCE POLICY

4.1 Minimizing the Cost Rate

As shown by Barlow and Hunter/1/, if one wishes to minimize the cost rate, ie, the cost per unit of operating of time which is given by

$$C(T) = \frac{C_M \bar{F}(t) + C_R F(T)}{\int_0^T \bar{F}(t)\, dt} \qquad (10)$$

the solution is the same as equation (9) with the mean durations m_R and m_M being replaced by the costs C_R and C_M respectively.

4.2 Maximizing Profit and Earning Rates

Handlarski/5/ proposed an approach where the optimum T is obtained to maximize the profit rate e(T) of the system. If e_o is the earnings per unit time when the system is operating, one has

$$e(T) = \frac{e_o \int_0^T \bar{F}(t)dt - C_M \bar{F}(T) - C_R F(T)}{\int_0^T \bar{F}(t)dt + m_M \bar{F}(T) + m_R F(T)} \qquad (11)$$

The optimum T satisfies the equation

$$h(T) \left[\int_0^T \bar{F}(t)dt + \frac{C_R m_M - C_M m_R}{e_o m_R + C_R - e_o m_M - C_M}\right] - F(T) = \frac{e_o m_R + C_M}{e_o m_R + C_R - e_o m_M - C_M} \qquad (12)$$

The model proposed by Nagasaki/7/ is a special case of the above with $m_R = m_M$. The above equation then becomes

$$h(T) \left[\int_0^T \bar{F}(t)dt + m_R\right] - F(T) = (e_o m_R + C_M)/(C_R - C_M) \qquad (13)$$

If the costs are proportional to the times, then the equation for optimum T becomes the same as equation (9), ie,

$$h(T)\int_0^T \bar{F}(t)dt - F(T) = m_M/(m_R - m_M) \qquad (14)$$

4.3 Minimal Repair

Beichelt and Fischer/3/ proposed a model where the system has two types of failure: one is removed by minimal repair, and the second, by a normal overhaul. The repair is said to be minimal, when after repair, the system is in the same state as before failure, ie, it is 'as bad as old'. Let p be the probability that a failure requires an overhaul.

It can be shown that an equivalent semi-Markov system is obtained when the hazard rate and mean repair time are replaced by p h(t) and $m_R + m_m(1-p)/p$ respectively where m_m is the average time to perform minimal repair.

The optimum T is thus given by

$$ph(T)\int_0^T \{F(t)\}^p\,dt - [1 - \{F(t)\}^p] = m_M/[m_R + m_m(1-p)/p - m_M] \qquad (15)$$

again the same form as equation 9.

Note that p = 1 corresponds to the standard age based maintenance policy and p = 0 corresponds to Barlow and Proschan's minimal repair policy/2/.

4.4 Imperfect Preventive Maintenance

Maintenance is termed imperfect when after maintenance, the system does not return to the 'as new' condition. Murthy and Nguyen/6/ studied a model where the system goes from state 'M' to state 'R' with a probability p_1, (ie, failure is due to preventive maintenance action). Such a transition is possible especially in old systems; for example, a corroded switch breaks when it is opened.

The unavailability is given by

$$U(T) = \frac{(m_M + p_1 m_R)\,\bar{F}(T) + m_R\,F(T)}{\int_0^T \bar{F}(t)dt + (m_M + p_1 m_R)\,\bar{F}(T) + m_R\,F(T)} \qquad (16)$$

The optimal T is given by

$$h(T) \int_0^T \bar{F}(t)dt - F(T) = (m_M + p_1 m_R)/(m_R - m_M - p_1 m_R) \quad (17)$$

again in the same form as equation (9).

5. A MODEL FOR MAINTENANCE DEPENDENT REPAIR DURATIONS

5.1 The Model

In the previous sections, it has been assumed that the mean repair duration is indepenent of the maintenance activities. This assumption may not always be valid, especially for power system components. For example, the severity of the failures may increase as the interval between successive maintenances is increased. That is, the mean repair duration $m_R(T)$ might be an increasing function of T. This section proposes a new model based on this assumption. Note that $m_R(T)$ is a deterministic function of T, the maintenance interval, and is assumed to be known. For a given T, the repair time is a random variable with a mean $m_R(T)$.

The availability of the system under consideration is given by

$$A(T) = \frac{\int_0^T \bar{F}(t)\ dt}{\int_0^T \bar{F}(t)\ dt + m_R(T)\ F(T) + m_M \bar{F}(T)} \quad (18)$$

The optimum maintenance interval is given by the solution of the equation

$$\left[h(T) + \frac{m_R'(T)\ F(T)}{(m_R(T) - m_M)\ \bar{F}(T)}\right] \int_0^T \bar{F}(t)\ dt - F(T)$$

$$= \frac{m_M}{m_R(T) - m_M} \quad (19)$$

where $m'_R(T)$ is the derivative of $m_R(T)$ with respect to T.

5.2 An Example

Consider the case where the mean repair time increases exponentially from m_{RO} to km_{RO} as T increases from 0 to ∞, ie, let $m_R(T) = km_{RO} - (k-1)\ m_{RO}\ e^{-T}$. This assumption seems

reasonable because it is likely that the repair time will not exceed an upper limit even if maintenance interval is increased beyond any large value.

Figures 6(a), 6(b) and 6(c) plot the optimum maintenance interval and the ratios N_R/N_M and U_R/U_M, for the case $k = 5$ when the time to failure distribution is Weibull. It should be noted that the horizontal axis is the ratio m_{RO}/m_M, that is, the ratio of the durations for $T = 0$.

Comparing these figures with Figures 3(b), 4(b) and 5(b) for the classical model, one can see that, as expected, the optimum interval is lower when the mean repair duration increases with T. It can also be seen that maintenance is effective even for $m_{RO}/m_M < 1$, unlike the earlier case (Figure 3(b)) where $T_{opt} = \infty$ for $m_R/m_M \leq 1$. Also, the ratio N_R/N_M is almost always <1, ie, the number of maintenance activities are higher than before (Figure 6(b)). The optimum unavailability due to maintenance is always more than that due to repair (Figure 6(c)).

6. CONCLUSIONS

The models and methods proposed in this study should be applicable to most power system components. The primary task will be to identify the failure distributions of the various components. In power systems, the optimum maintenance policy of a component will also depend on the characteristics of the neighbouring components. Extensions of the above methods to these cases should be investigated.

A detailed study of the effect of maintenance interval on the repair duration should prove very useful. Also, repairs which can be classified as minimal should be identified and the relevant probability that a failure needs only a minimal repair should be estimated from historical records. These problems will be studied in future investigations.

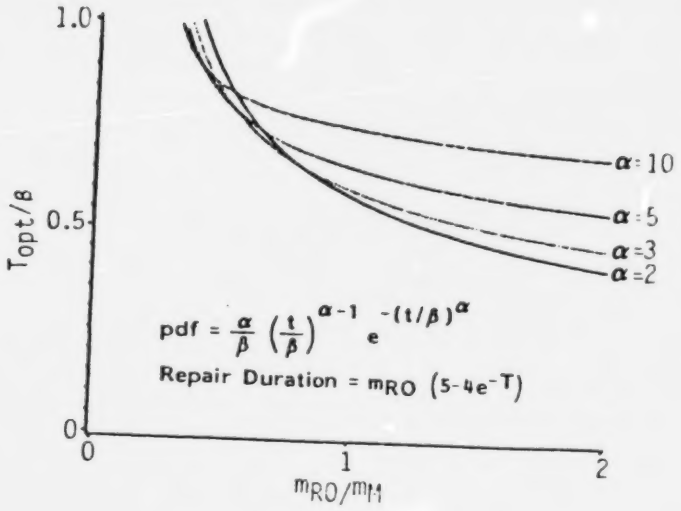
Approved:

J.G. Cassan
Manager
Operations Research Dept

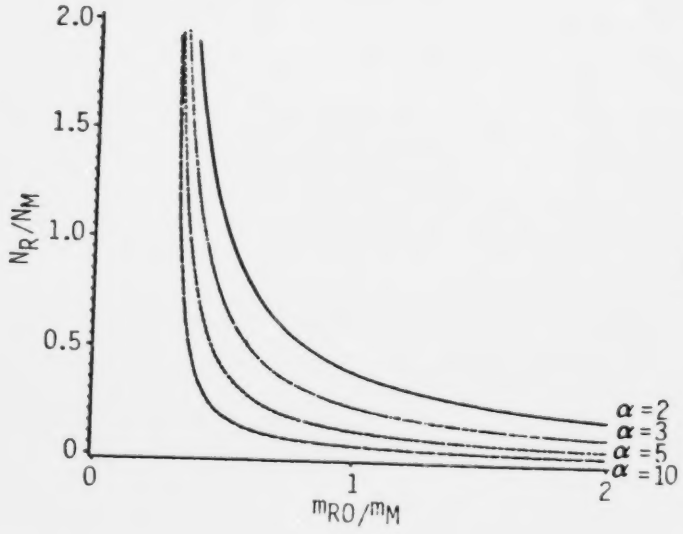
Submitted:

N. Ramani
Engineer
Reliability and Statistics Section

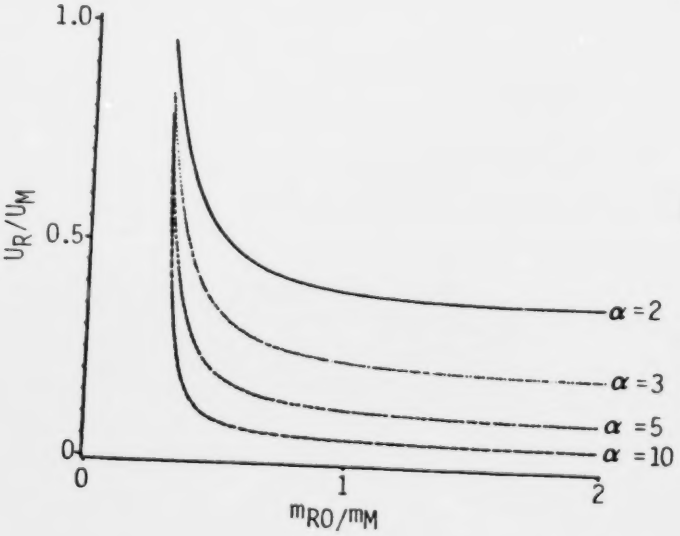
NR:djb/kk



(a)
Optimum Maintenance Interval



(b)
Ratio of Repair to Maintenance Number of Events



(c)
Ratio of Repair to Maintenance Unavailability

FIGURE 6
EXAMPLE ILLUSTRATING A
MAINTENANCE DEPENDENT REPAIR DURATION MODEL

REFERENCES

1. Barlow, R. and Hunter, L. "Optimum Preventive Maintenance Policies", Operations Research, Vol 8, No 1, pp 90-100, (1960).

2. Barlow, R. and Proschan, F. "Mathematical Theory of Reliability", New York, Wiley, (1965).

3. Beichelt, F. and Fischer, K. "General Failure Model Applied to Preventive Maintenance Policies", IEEE Transactions on Reliability, Vol R-29, No 1, pp 39-41, (1980).

4. Cox, D.R. and Miller, H.D. "Theory of Stochastic Processes", Chapman and Hall, (1965).

5. Handlarski, J. "Mathematical Analysis of Preventive Maintenance Schemes", Journal of Operational Research Society, Vol 31, No 3, pp 227-237, (1980).

6. Murthy, D.N.P. and Nguyen, D.G. "Optimal Age-Policy with Imperfect Preventive Maintenance", Vol R-30, No 1, pp 80-81, (1981).

7. Nakagawa, T. "Optimum Preventive Maintenance Policies for Repairable System", IEEE Transactions on Reliability, Vol R-26, No 3, pp 168-173, (1977).

8. Pyke, R. "Markov Renewal Processes: Definitions and Preliminary Properties", Annals of Mathematical Statistics, Vol 32, No 4, pp 1231-1242, (1961).

9. Pyke, R. "Markov Renewal Processes with Finitely Many States", Annals of Mathematical Statistics, Vol 32, No 4, pp 1243-1259, (1961).

10. Ramani, N. "Optimum Maintenance Policies - A Literature Survey", Ontario Hydro Research Division Report No 83-408-K, (1983).

APPENDIX A

SEMI-MARKOV PROCESSES

This Appendix gives a brief review of semi-Markov processes. A detailed account can be found elsewhere/2,4,8,9/.

Consider a physical system which can be in one of k states and which is observed at every instant when a state change occurs. Let the successive observations (states of the system) be denoted by X_0, X_1 ... X_n. X_n is a discrete random variable and the sequence $\{X_n\}$ is a Markov chain if the system is memoryless, ie, for any real numbers x_0, x_1 ... x_n, one has

$$P[X_n = x_n | X_0 = x_0, \ \ldots \ X_{n-1} = x_{n-1}] = P[X_n = x_n | X_{n-1} = x_{n-1}] \tag{A.1}$$

The Markov chain is homogeneous if the transition probabilities do not depend on n, ie, if

$$P[X_n = j | X_{n-1} = i] = P_{ij} = \text{a constant} \tag{A.2}$$

Note that $P_{ij} \geq 0$ for all i, j

and $\sum_j P_{ij} = 1$ for all i

Now define a stochastic process $\{z(t), t > 0\}$ such that $z(t) = i$ denotes that the process is in state i at time t. The process is characterized by two matrices:

1) $[P] = [P_{ij}]$, the k x k transition probability matrix of the Markov chain

and

2) the distribution function matrix $[F] = [F_{ij}(t)]$

   where $F_{ij}(t)$ is the cumulative distribution of the waiting time in state i, given that the next transition will be to state j.

If all the waiting time distributions are exponential, the process becomes a continuous time homogenous Markov process, ie, for any real numbers, x_1, ... x_n, and any set of time points $t_1 < t_2 \ldots < t_n$, one has

$$P\left[X(t_n) \leq x_n \middle| x(t) = x_1, \ldots X(t_{n-1}) = x_{n-1}\right]$$

$$= P\left[X(t_n) \leq x_n \middle| X(t_{n-1}) = x_{n-1}\right] \qquad (A.3)$$

For general waiting time distributions, the process is Markovian only at the instants of time at which the transitions take place and the process is known as a semi-Markov process. The original Markov chain is said to be 'embedded' in this process.

Let

$$p_{ij}(t) = P[x(t) = j \mid z(0) = i] \qquad (A.4)$$

= probability that the system is in state j at time t given that it is in state i at time 0.

It can be shown/4/ that

$$s[P^*(s)] = [I - g^*(s)]^{-1} [P - g^*(s)] \qquad (A.5)$$

where

$$[p(t)] = [p_{ij}(t)],$$

$$[g(t)] = [p_{ij}\ f_{ij}(t)],$$

$f^*(s)$ denotes the Laplace transform of the function $f(t)$ and s is the Laplace operator

For transient solutions, both [P] and [F] are needed but steady-state results can be obtained from [P] and the mean vector $[\mu]$ where μ_i is the mean residence time in state i. μ_i is given by $E[T_i]$ where T_i is the time spent in state i and the distribution of T_i is

$$\sum_j p_{ij}\ F_{ij}.$$

Let π_i be the steady-state probability that the embedded Markov chain is in state i. It can be shown /4/ that $[\pi] = [\pi_i]$ is given by the solution of

$$\pi P = \pi \qquad (A.6)$$

with $\sum_i \pi_i = 1$. The steady-state probability that the semi-Markov process is in state i, is given by

$$p_i = \pi_i \mu_i / \sum_j \pi_j \mu_j \qquad \text{(A.7)}$$

For the purpose of this study, it should be noted that, any system where the starting instant of each state is a regeneration point, ie time can be reset to zero, can be analyzed using the semi-Markov approach.

APPENDIX B

SUM-OF-EXPONENTIALS DISTRIBUTIONS

For complex systems, it is rarely, if ever, possible to determine the distribution of the time to failure and one often resorts to the use of empirical distributions. An interesting case is where the probability density is represented by a sum of exponential functions. Consider a reliability function of the type

$$\bar{F}(t) = a_1 e^{-\lambda t} + a_2 e^{-2\lambda t} \qquad (B.1)$$

$$a_1 > 0$$

$$a_1 + a_2 = 1$$

The main advantage of using such a distribution is that equation (9) becomes a polynomial in $e^{-\lambda T}$ and thus the solution is fairly straightforward. It can be shown that the above expression represents a true density function for $0 \leq a_1 \leq 2$. Also, the hazard state is decreasing for $a_1 < 1$ and is increasing for $a_1 > 1$. The above can be generalized to expressions of the form

$$\bar{F}(t) = a_1 e^{-\lambda t} + a_2 e^{-\alpha\lambda t} \qquad (B.2)$$

$$\alpha > 1$$

$$a_1 > 0$$

$$a_1 + a_2 = 1$$

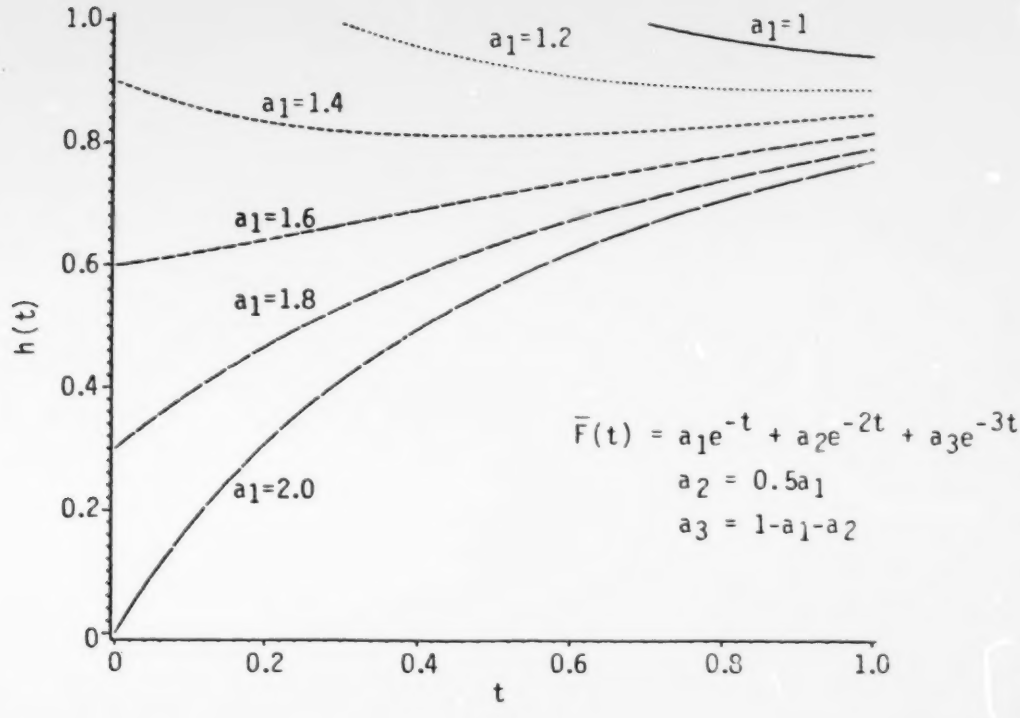
and

$$\bar{F}(t) = \sum_i a_i e^{-i\lambda t} \qquad (B.3)$$

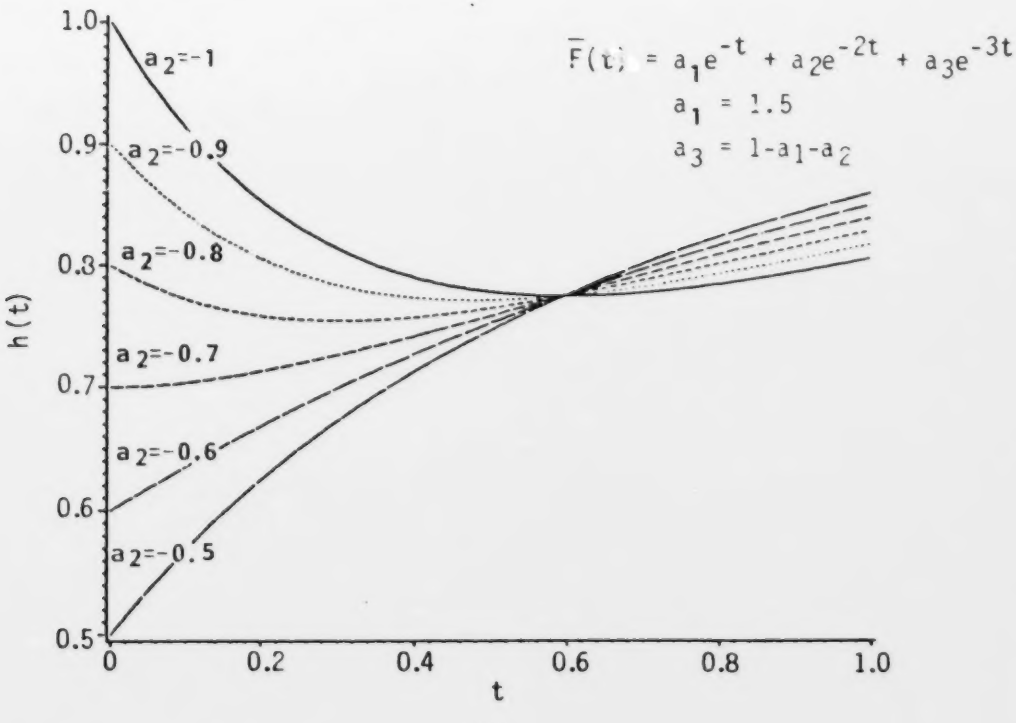
$$a_1 > 0$$

$$\sum a_i = 1$$

The distribution represented by equation (B.2) with $\alpha = a_2/a_1$ is often referred to in the literature as the hyper-exponential distribution. It should be noted that the above expressions do not always represent physical systems with series/parallel stages. They are intended primarily as empirical distributions which can be fit to field data using methods such as moment matching. They can fit several gamma type distributions where the hazard rates approach a finite values as $t \to \infty$. Figures 7(a) and 7(b) show the plots of the hazard rates for some values of the parameters.



(a)



(b)

FIGURE 7

HAZARD RATES FOR SUM OF EXPONENTIALS DISTRIBUTIONS